# CHAINS COME FIRST

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## 1 An introduction to $k$-measures

When closing the second chapter (Vector fields and differential forms) of his handy introduction to differential geometry [1], our friend and mentor Alain Bossavit regrets that

> [s]ome dissymmetry has crept in with this proliferation of covectors, with the effect to spoil the simple vector-covector duality we had at the beginning. One regains balance by introducing objects dual to the $p$-covectors for $p > 1$. One thus calls $p$-vector an element of the dual of the vector space of $p$-covectors. [ . . . ] The Grassmann algebra of multi-vectors also exists, but is less popular and less often applied than the multi-covectors one.

I, in turn, find regrettable the persistent imbalance between the notion of $p$-covector *fields*—ubiquitous, hence worth the terse nickname of *$p$-forms*—and that of *$p$-vector* fields—seldom, if ever, mentioned. It has taken me a long time, but just recently I have finally been able to restore perfect symmetry, after rediscovering the key notion of *$k$-measures* in a rather old and little-known paper [2] by the well-know Italian mathematician Gaetano Fichera.

Basically, a $k$-measure on a $n$-dimensional manifold $\mathcal{M}$ is a $k$-vector valued measure. Any such measure may be decomposed in a unique way into the sum of a singular $k$-measure and a $k$-measure absolutely continuous with respect to the canonical measure on $\mathcal{M}$, defined on each chart domain $\mathcal{U} \subset \mathcal{M}$ as the pull-back of the Lebesgue measure on $\mathbb{R}^n$ under the coordinate chart $\varphi : \mathcal{U} \to \mathbb{R}^n$. Absolutely continuous $k$-measures may be identified with locally summable $k$-vector fields. The space $\mathfrak{M}_k$ of $k$-measures is the Banach dual of the space $\mathbf{\Lambda}_k^0$ of *continuous* $k$-forms, the duality pairing between $\mu \in \mathfrak{M}_k$ and $\psi \in \mathbf{\Lambda}_k^0$ being represented by the integral of $\psi$ with respect to $\mu$. Any $k$-submanifold $\mathcal{S} \subset \mathcal{M}$ ($k \leq n$) induces a distinguished $k$-measure $\mu_{\mathcal{S}}$ supported by the closure of $\mathcal{S}$—hence singular, unless $k = n$. In Section 2, I will take advantage of the existence of *absolutely continuous* $k$-measures for all $k < n$.

On this basis, I am able to define in a satisfactory way the *boundary* of $k$-measures in terms of Lie derivatives, paraphrasing Palais [3]. The exterior derivative of $(k-1)$-forms emerges naturally via duality. If $\mu \in \mathfrak{M}_k$ is not too singular, its boundary $\partial_k \mu$ is in $\mathfrak{M}_{k-1}$. In particular, the boundary of the $k$-measure $\mu_{\mathcal{S}}$ induced by a $k$-submanifold with boundary $\mathcal{S}$ is the $(k-1)$-measure induced by its boundary $\partial\mathcal{S}$:

$$\partial_k \mu_{\mathcal{S}} = \mu_{\partial\mathcal{S}} \,.$$

## 2 From measured to metrized chains

The spaces of real-valued $k$-chains and $k$-cochains are the discrete analogs—or, better, antecedents—of the spaces of $k$-measures and $k$-forms, respectively. Chains come first, with the related hierarchy of boundary operators; *co*chains and *co*boundary follow. Cochains represent densities with respect to the measures imparted to cells by chains, and the duality pairing between them is a discrete preliminary to integration.

Measure, however, does not exhaust geometry. To impart *metric* properties to the space of chains, one has to endow it with a Euclidean inner product. Since the metric structure of the underlying manifold $\mathcal{M}$ carries basic information on the physical phenomena taking place on that scene, it is vital that this structure be properly mimicked by discrete geometric models meant to be used for trustworthy physics-based simulations. Following and perfecting [4], I associate linearly an absolutely continuous and square-integrable $k$-measure on $\mathcal{M}$ with each $k$-chain, and identify the inner product between two $k$-chains with the inner product between the corresponding $k$-vector fields.

The space of $k$-chains has a *standard* basis, formed by *unit* $k$-chains, i.e., chains attaching the unit value to a single $k$-cell and the null value to all the others; the standard basis of the space of $k$-cochains is its dual. The *trivial inner product* makes the standard bases *orthonormal*, identifying each unit chain with the corresponding unit cochain. Such an assumption, while expedient to use on any given cell complex, is totally unrelated—in general—to the geometric properties relevant to the physical phenomena taking place on the underlying manifold $\mathcal{M}$. The seemingly obvious identification between unit chains and cochains is, on the contrary, a recipe for disaster, since identifying properly chains with cochains is essential for importing the relevant, physics-based metric structure into the discrete model. Only by doing so boundary and coboundary operators may be composed with each other giving rise to physically meaningful Laplace-deRham operators. This issue is also basic to gain the possibility of a proper information transfer from a cell complex to any of its refinements (and vice versa), and to establish a notion of convergence for refinement sequences.

Multivector fields associated with chains should satisfy the following conditions: i) *representativeness*: the $k$-vector field associated with a unit $k$-chain—*smearing* field, for short—is tangent to the underlying $k$-cell; ii) *locality*: the support of each smearing field contains the corresponding $k$-cell and is contained in the union of the $n$-cells intersecting it; iii) *square-integrability*: each smearing field is square-integrable; iv) *asymptotic completeness*: in the limit of infinite mesh refinement, all locally square-integrable $k$-measures are approximated by linear combinations of the smearing fields.

## References


[1] A. Bossavit, *Differential Geometry for the Student of Numerical Methods in Electromagnetism* (1991) `http://butler.cc.tut.fi/ bossavit/`.

[2] G. Fichera, *Spazi lineari di k-misure e di forme differenziali*, Proceedings of the International Symposium on Linear Spaces held at the Hebrew University of Jerusalem, July 5–12, 1960 (1961), 175–226.

[3] R.S. Palais, *A definition of the exterior derivative in terms of Lie derivatives*, Proceedings of the American Mathematical Society **5** (1954), 902–908.

[4] A. DiCarlo, F. Milicchio, A. Paoluzzi, and V. Shapiro, *Discrete physics using metrized chains*, Symposium on Solid and Physical Modeling (W.F. Bronsvoort et al., eds.), ACM (2009), 135–145.