Whitney Form Discretization of Helicity Functionals and Applications

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Abstract

Although explicit Whitney-form finite element discretizations of helicity functionals have been in the engineering literature for over two decades, there has been little connection between the practice of finite elements and the original mathematical motivation for developing Whitney forms. For the helicity functional, these discretizations exhibit a discrete version of the metric invariance enjoyed by the functional defined in the continuum, and have applications to geometric inverse problems such as impedance tomography and the synthesis of force-free magnetic fields.

The current paper focuses on simplicial meshes arising from locally logically rectangular meshes and reveals some surprising ties between the spectrum of the operator and topological invariants of the mesh. In particular, it reveals spectral quantities which are topological obstructions to having a globally logically rectangular mesh. Although the square of the helicity operator is intimately related to the Laplacian-Beltrami operator, the discetization of the Laplacian yields no insights into the topological invariants we develop.

Although these topological invariants, related to spectral asymmetry of the discretized helicity functional, have no obvious connection to to the discretized Laplace-Beltrami operator, the techniques used in this presentation are ultimately related to those which arise in the analysis of the graph Laplacian and adjacency matrix of a graph. The subtlety is that the underlying elimination graph in question has nodes which are associated with the edges of the underlying tetrahedral finite element mesh, and each element in the mesh gives rise to three edges in the elimination graph. The anaysis is more subtle than usual in that the elimination graph is oriented, and the nonzero entries of the adjacency matrix aquire signs.

Along the way we construct logically rectangular meshes for which the elimination graph falls into three disconnected components. For any fixed geometric model, the topological obstructions to having a logically rectangular mesh developed in this paper are then tied to obstructions to having an elimation graph which consists of three separate connected components.